Journal of Multivariate Analysis **73**, 241–261 (2000)
doi:10.1006/jmva.1999.1880, available online at http://www.idealibrary.com on IDEAL®

Limit Laws for Symmetric k-Tensors of Regularly Varying Measures

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Revised June 24, 1998

We consider the asymptotics of certain symmetric k-tensors, the vector analogue of sample moments for i.i.d. random variables. The limiting distribution is operator stable as an element of the vector space of real symmetric k-tensors.

AMS 1991 subject classifications: Primary 60F05; Secondary 60E07.
Key words and phrases: regularly varying measures, domains of attraction, operator stable laws, symmetric tensors.

1. INTRODUCTION

In this paper we establish the asymptotics of certain symmetric k-tensors whose underlying distribution is regularly varying. Regular variation is an asymptotic property of probability measures with heavy tails. Regular variation describes the power law behavior of the tails. Tensors and tensor products are useful in probability and statistics; see, for example, [8, 15, 19]. Random tensors are considered in [7, 12, 14]. The most familiar example is the sample covariance matrix, which is a random 2-tensor. The tensors considered here are also the vector analogue of sample moments for i.i.d. random variables, and in this sense our results extend those of [6, 17, 20].

Regularly varying probability measures on finite dimensional real vector spaces are of special interest in analyzing the generalized central limit theorem. Let $X, X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.)

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random vectors with common distribution μ on $\mathbb{R}^d$. If there exist linear operators A_n and nonrandom vectors b_n such that

$$A_n(X_1 + \cdots + X_n) - b_n \Rightarrow Y \tag{1.1}$$

as $n \to \infty$ for some full (that is not supported on any proper hyperplane) random vector Y we say that X belongs to the generalized domain of attraction of Y and write $X \in \mathrm{GDOA}(Y)$. Here $\Rightarrow$ denotes convergence in distribution.

Due to a result of Sharpe [18] the distribution ν of Y is operator stable, that is ν is infinitely divisible and there exists a linear operator E called an exponent of ν and shifts $a_t \in \mathbb{R}^d$ such that

$$\nu^t = t^E \nu * \delta(a_t) \qquad \text{for all} \quad t > 0. \tag{1.2}$$

Here ν^t is the tth convolution power of ν, $t^E = \exp(E \log t)$, $\delta(a)$ denotes the point mass in $a \in \mathbb{R}^d$, $*$ denotes convolution and $(A\nu)(B) = \nu(A^{-1}B)$ for Borel sets $B \subset \mathbb{R}^d$. We call ν $(t^E)_{t>0}$-operator stable if 1.2 holds.

If ν has no normal component GDOA(Y) was characterized by Meerschaert using a multivariable theory of regular variation. In [10, 11] it is shown that $X \in \mathrm{GDOA}(Y)$ if and only if μ is regularly varying with index E, where E is as in (1.2) This means that there exists a sequence (A_n) of linear operators with $A_{[\lambda n]} A_n^{-1} \to \lambda^{-E}$ for all $\lambda > 0$, (we say that (A_n) is regularly varying with index $-E$) and a σ-finite measure ϕ on $\mathbb{R}^d \setminus \{0\}$ which is finite outside every neighborhood of the origin and which is not supported on any proper subspace such that

$$n \cdot (A_n \mu) \to \phi \qquad \text{as} \quad n \to \infty. \tag{1.3}$$

The convergence in (1.3) is $n\mu(A_n^{-1}B) \to \phi(B)$ as $n \to \infty$ for all Borel subsets $B \subset \mathbb{R}^d \setminus \{0\}$ which are bounded away from the origin and whose topological boundary has ϕ-measure zero. In this case we write $\mu \in \mathrm{RV}(E)$. For more information about regularly varying sequences of linear operators and regularly varying measures see [9, 13].

It turns out that in this case the measure ϕ in (1.3) is the Lévy measure of ν. Furthermore it follows from [18] that the real parts of the eigenvalues of E are greater than 1/2.

To illustrate our results let us consider the simple one dimensional case. Assume that $Z_1, Z_2, \ldots$ are i.i.d. symmetric random variables such that the tail function $V_0(t) = P\{|X_1| > t\}$ is regularly varying with index $-\rho$ for some $\rho > 0$. That is $V_0(\lambda t)/V_0(t) \to \lambda^{-\rho}$ as $t \to \infty$. It is a classical result (see, e.g., [2]) that if $0 < \rho < 2$ then there exist $a_n > 0$ such that

$$a_n(Z_1 + \cdots + Z_n) \Rightarrow G_\rho \qquad \text{as} \quad n \to \infty,$$

where G_ρ is a ρ-stable random variable on $\mathbb{R}$. Furthermore, if $\rho > 2$ then $EZ_1^2 < \infty$ and hence the central limit theorem applies.

Let now $\rho > 2$ and choose a natural number $k \geqslant 1$ such that $2k > \rho$. We want to investigate the asymptotic behavior of $Z_1^k + \cdots + Z_n^k$. Since $P\{|Z_1^k| > t\} = V_0(t^{1/k}) = \bar{V}_0(t)$ and $\bar{V}_0$ is regularly varying with index $-\rho/k$, where $0 < \rho/k < 2$ it follows that there exist $r_n > 0$ such that

$$r_n(Z_1^k + \cdots + Z_n^k) \Rightarrow G_{\rho/k} \qquad \text{as} \quad n \to \infty.$$

The purpose of this work is to generalize this concept to the multivariable case. The natural analog of taking powers of random variables in the vector valued case is tensor products $\bigotimes^k X_i$ of a random vector X_i on $\mathbb{R}^d$. Hence we are looking at the asymptotic distribution of normalized sums of i.i.d. symmetric k-tensors $\bigotimes^k X_i$. In [12, 14] we consider the most familiar case $k=2$, which is the sample covariance matrix. This paper extends the asymptotic results of those papers to more general k-tensors.

We also investigate the structure of the limit distributions. Our main results are then applied to construct an estimate of the biggest real part of the eigenvalues of E from an i.i.d. sample with distribution $\mu \in \mathrm{RV}(E)$, which plays a crucial role in the multivariable theory of regular variation. Furthermore we prove a stochastic compactness result for certain polynomials of the components of the X_i.

2. TENSOR ALGEBRA, NOTATION AND PRELIMINARY RESULTS

In this section we introduce the vector space of symmetric k-tensors as well as some notation necessary for our main results. Furthermore we state some of the multivariate regular variation theory which will be crucial in the proof of our limit theorem.

Let $d \geqslant 1$ and $k \geqslant 2$ be natural numbers and let $\mathbb{V} = (\mathbb{R}^d \otimes \cdots \otimes \mathbb{R}^d)_s$ denote the real vector space of symmetric k-tensors on $\mathbb{R}^d$. If we define $T\colon \mathbb{R}^d \to \mathbb{V}$; $T(x) = x \otimes \cdots \otimes x = \bigotimes^k x$ then $\mathbb{V}$ is the linear span of $T(\mathbb{R}^d)$.

For $x, y \in \mathbb{R}^d$ we define $\langle T(x), T(y) \rangle = \langle \bigotimes^k x, \bigotimes^k y \rangle = \langle x, y \rangle^k$. Extend by linearity to define an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{V}$. Then $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ is an Euclidean vector space and $\|M\| = \langle M, M \rangle^{1/2}$ is the corresponding norm.

For a linear operator A on $\mathbb{R}^d$ let $L_A\colon \mathbb{V} \to \mathbb{V}$ be the linear extension of $L_A(\bigotimes^k x) = \bigotimes^k (Ax)$. The following properties are obvious:

Let A be an invertible operator on $\mathbb{R}^d$, $x \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$ and $M, N \in \mathbb{V}$. Then

(a) $L_{A^{-1}} = (L_A)^{-1}$,

(b) $L_A(T(x)) = T(Ax)$,

(c) $T(\lambda x)=\lambda^k T(x)$,

(d) $\langle L_A(M), N\rangle = \langle M, L_{A^*}(N)\rangle$,

where A^* denotes the transpose of A. For more details on tensor algebra see, e.g., [3].

Now let $\mu \in M^1(\mathbb{R}^d)$ be a probability measure. We say that μ is regularly varying with index E if there exists a sequence (A_n) of linear operators with $A_{[\lambda n]}A_n^{-1} \to \lambda^{-E}$ and if there exists a σ-finite measure ϕ on $\Gamma=\mathbb{R}^d\setminus\{0\}$ which is finite outside every neighborhood of the origin such that

$$n(A_n\mu)\to\phi \qquad \text{as} \quad n\to\infty. \tag{2.1}$$

It follows (see [9]) that

$$t\cdot\phi=(t^E\phi) \qquad \text{for all} \quad t>0. \tag{2.2}$$

We write $\mu\in \mathrm{RV}(E)$. In what follows let $a_1<\cdots<a_p$ denote the real parts of the eigenvalues of E. Note that by [13] we necessarily have $a_1\geqslant 0$.

Regular variation of a probability measure μ implies certain bounds on the tail function as well as the truncated moment functions of μ. For a unit vector $\theta\in\mathbb{R}^d$, $t>0$, and $\eta, \zeta\geqslant 0$ let

$$V_\eta(t,\theta)=\int_{|\langle x,\theta\rangle|>t} |\langle x,\theta\rangle|^\eta \, d\mu(x)$$

and

$$U_\zeta(t,\theta)=\int_{|\langle x,\theta\rangle|\leqslant t} |\langle x,\theta\rangle|^\zeta \, d\mu(x)$$

denote the tail and truncated moment function of μ in the direction θ, respectively, whenever V_η exists.

Now, if $\mu\in\mathrm{RV}(E)$ and $0<a_1<\cdots<a_p$ are the real parts of the eigenvalues of E, we get from Lemma 2 of [10] that if $\eta<1/a_p$ then for all $\delta>0$ there exist constants $m, r_0>0$ such that

$$\frac{V_\eta(\lambda r,\theta)}{V_\eta(r,\theta)}\geqslant m\lambda^{-1/a_1+\eta-\delta} \tag{2.3}$$

for all $\|\theta\|=1$, $\lambda\geqslant 1$, and $r\geqslant r_0$. Furthermore, if $\zeta>1/a_1$ then for every $\delta>0$ there exist constants $M, r_1>0$ such that

$$\frac{U_\zeta(\lambda r,\theta)}{U_\zeta(r,\theta)}\leqslant M\lambda^{-1/a_p+\zeta+\delta} \tag{2.4}$$

for all $\|\theta\|=1$, $\lambda\geqslant 1$, and $r\geqslant r_0$.

Finally we need a uniform version of an exercise in [2, p. 289] which follows along the same lines as the one-dimensional result: For $\zeta > 1/a_1$ and $\eta < 1/a_p$ there exist positive constants A, B, t_0 such that

$$A \leqslant \frac{t^{\zeta-\eta} V_\eta(t, \theta)}{U_\zeta(t, \theta)} \leqslant B \tag{2.5}$$

for all $\|\theta\| = 1$ and $t \geqslant t_0$.

Remark 2.1. (a) By [10, 11] we have

$$A_n \mu^{*n} * \delta(-b_n) \Rightarrow \nu,$$

where ν has no normal component if and only if $\mu \in \mathrm{RV}(E)$ and $a_1 > 1/2$.

(b) It follows from Lemma 2 of [10] that if $\mu \in \mathrm{RV}(E)$ and $a_p < 1/2$ then $\int x^2 \, d\mu(x) < \infty$ and hence the central limit theorem applies.

Recall that an infinitely divisible probability measure μ on a finite dimensional real vector space $\mathbb{B}$ is characterized by the Lévy–Khinchin formula. That is, μ is infinitely divisible if and only if its Fourier transform $\hat{\mu}$ can be written as

$$\hat{\mu}(y) = \exp\left\{ i\langle c, y\rangle - \frac{1}{2} Q(y) + \int_{\mathbb{B}\setminus\{0\}} \left(e^{i\langle x, y\rangle} - 1 - \frac{i\langle x, y\rangle}{1 + \|x\|^2} \right) d\varphi(x) \right\},$$

where $c \in \mathbb{B}$, Q is a positive semidefinite quadratic form on $\mathbb{B}$ and φ is a σ-finite measure on $\mathbb{B}\setminus\{0\}$ which satisfies

$$\int_{\mathbb{B}\setminus\{0\}} \min(\|x\|^2, 1) \, d\varphi(x) < \infty.$$

φ is called the Lévy measure of μ and we say that μ has Lévy representation $[c, Q, \varphi]$.

3. MAIN RESULTS

Suppose that $X_1, X_2, \ldots$ are i.i.d. random vectors whose common distribution μ is regularly varying. In this section we prove that symmetric k-tensors of the form

$$M_n = \sum_{i=1}^{n} \bigotimes^k X_i$$

are asymptotically operator stable, for all k sufficiently large. Our main technical tools are regular variation, along with the standard convergence criteria for triangular arrays.

THEOREM 3.1. *Suppose that $X_1, X_2, \ldots$ are i.i.d. random vectors with common distribution $\mu \in \mathrm{RV}(E)$ and that (2.1) holds. For any natural number $k \geqslant 2$ such that $1/(2k) < a_1$ there exist nonrandom $B_n \in \mathbb{V}$ such that*

$$L_{A_n}(M_n) - B_n \Rightarrow W, \tag{3.1}$$

where W is infinitely divisible on $\mathbb{V}$ with Lévy representation $[C, 0, T(\phi)]$ and ϕ is the limit measure in (2.1.)

Proof. First we show that $T(\phi)$ is a Lévy measure on $\mathbb{V}$. Since $T(\phi)$ is a σ finite measure on $\mathbb{V}$ which is finite outside every neighborhood of the origin, all we have to show is that

$$\int_{\mathbb{V}\setminus\{0\}} \min(\|A\|^2, 1)\, dT(\phi)(A) < \infty. \tag{3.2}$$

Since $\|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle x, x\rangle^k = \|x\|^{2k}$ we have

$$\begin{aligned}\int_{\mathbb{V}\setminus\{0\}} \min(\|A\|^2, 1)\, dT(\phi)(A) &= \int_{0<\|A\|\leqslant 1} \|A\|^2\, dT(\phi)(A)\\ &\quad + T(\phi)\{A \in \mathbb{V} : \|A\| > 1\}\\ &= \int_{0<\|x\|\leqslant 1} \|x\|^{2k}\, d\phi(x) + \phi\{x : \|x\| > 1\}.\end{aligned}$$

But $\phi\{x: \|x\| > 1\} < \infty$ so we have to show that

$$\int_{0<\|x\|\leqslant 1} \|x\|^{2k}\, d\phi(x) < \infty. \tag{3.3}$$

By estimates in [4, Chapter 6] it follows that for every $\delta > 0$ there exists a $M > 0$ such that $\|t^E\| \leqslant M t^{a_1 - \delta}$ for all $0 < t \leqslant 1$. Now fix any $c > 1$ and let $Q = \{x \in \mathbb{R}^d : a \leqslant \|x\| \leqslant b\}$ where $0 < a < b$ are chosen so that $\{x \in \mathbb{R}^d : 0 < \|x\| \leqslant 1\} \subset \bigcup_{\ell=0}^{\infty} c^{-\ell E}(Q)$. Choose $\delta > 0$ such that $1/(2k) < a_1 - \delta < a_1$. Then (2.2) implies

$$\begin{aligned}
\int_{0<\|x\|\leqslant 1} \|x\|^{2k}\, d\phi(x) &\leqslant \sum_{\ell=0}^{\infty} \int_{c^{-\ell E}(Q)} \|x\|^{2k}\, d\phi(x) \\
&\leqslant \sum_{\ell=0}^{\infty} \int_{c^{-\ell E}(Q)} \|c^{-\ell E}\|^{2k}\, \|c^{\ell E}x\|^{2k}\, d\phi(x) \\
&\leqslant M \sum_{\ell=0}^{\infty} c^{-\ell(a_1-\delta)\,2k} \int_Q \|x\|^{2k}\, d(c^{\ell E}\phi)(x) \\
&= M \sum_{\ell=0}^{\infty} c^{\ell(1-(a_1-\delta)\,2k)} \int_Q \|x\|^{2k}\, d\phi(x) < \infty,
\end{aligned}$$

since $1-(a_1-\delta)\,2k<0$ and $\int_Q \|x\|^{2k}\, d\phi(x)<\infty$. This proves (3.3) and hence by (3.2) $T(\phi)$ is a Lévy measure on $\mathbb{V}$.

We show (3.1) by applying the standard convergence criteria for triangular arrays; see [16]. That is, we have to show the following:

(a) For all Borel sets $S\subset \mathbb{V}\backslash\{0\}$ which are bounded away from the origin and whose topological boundary has $T(\phi)$ measure zero we have

$$nL_{A_n}(T(\mu))(S) \to T(\phi)(S). \tag{3.4}$$

(b) For all nonzero $B\in\mathbb{V}$ we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\ \limsup_{n\to\infty}\ n \Bigg[&\int_{\|A\|\leqslant\varepsilon} \langle A, B\rangle^2\, dL_{A_n}(T(\mu))(A) \\
&- \left(\int_{\|A\|\leqslant\varepsilon} \langle A, B\rangle\, dL_{A_n}(T(\mu))(A) \right)^2 \Bigg] = 0.
\end{aligned} \tag{3.5}$$

We first show (a). Let ∂S denote the topological boundary of a set S. Since T is continuous it follows that $\partial T^{-1}(S)\subset T^{-1}(\partial S)$. Hence if S is a $T(\phi)$-continuity set, $\phi(\partial T^{-1}(S))\leqslant \phi(T^{-1}(\partial S)) = T(\phi)(\partial S)=0$ and hence $T^{-1}(S)$ is a ϕ-continuity set. Moreover it is easy to see that if S is bounded away from the origin in $\mathbb{V}$ then $T^{-1}(S)$ is bounded away from the origin in $\mathbb{R}^d$. Then for such S we get

$$nL_{A_n}(T(\mu))(S) = nT(A_n\mu)(S) = n(A_n\mu)(T^{-1}(S)) \to \phi(T^{-1}(S)) = T(\phi)(S)$$

as $n\to\infty$, showing (3.4)

For the proof of (3.5) an application of the Cauchy–Schwarz inequality shows that it is enough to verify

$$\lim_{\varepsilon\to 0}\ \limsup_{n\to\infty}\ n \int_{\|A\|\leqslant\varepsilon} \langle A, B\rangle^2\, dL_{A_n}(T(\mu))(A) = 0 \tag{3.6}$$

for all unit vectors $B\in\mathbb{V}$.

Suppose first that $B = T(b) = \bigotimes^k b$ for some unit vector $b \in \mathbb{R}^d$. Then

$$\begin{aligned}
n \int_{\|A\| \leqslant \varepsilon} & \langle A, B \rangle^2 \, dL_{A_n}(T(\mu))(A) \\
& \leqslant n \int_{|\langle A, B\rangle| \leqslant \varepsilon} \langle A, B \rangle^2 \, dL_{A_n}(T(\mu))(A) \\
& = n \int_{|\langle L_{A_n}(T(x)), T(b)\rangle| \leqslant \varepsilon} \langle L_{A_n}(T(x)), T(b) \rangle^2 \, d\mu(x) \\
& = n \int_{|\langle T(A_n x), T(b)\rangle| \leqslant \varepsilon} \langle T(A_n x), T(b) \rangle^2 \, d\mu(x) \\
& = n \int_{|\langle x, A_n^* b\rangle| \leqslant \varepsilon^{1/k}} \langle x, A_n^* b \rangle^{2k} \, d\mu(x).
\end{aligned}$$

If we write $A_n^* b = r_n \theta_n$ for some $r_n > 0$ and $\|\theta_n\| = 1$ and use the definition of the function U_r above we get

$$n \int_{\|A\| \leqslant \varepsilon} \langle A, B \rangle^2 \, dL_{A_n}(T(\mu))(A) \leqslant n r_n^{2k} \, U_{2k}(r_n^{-1} \varepsilon^{1/k}, \theta_n).$$

Note that since $(A_n) \in \mathrm{RV}(-E)$ and $a_1 > 0$ we have $r_n \to 0$ as $n \to \infty$. Choose $\delta > 0$ such that $2 - (1/k)((1/a_1) + \delta) > 0$. Then, by (2.3) there exists a constant $m > 0$ such that for any fixed $0 < \varepsilon < 1$ there exists a $n_0 \geqslant 1$ such that for all $n \geqslant n_0$

$$\begin{aligned}
\frac{V_0(\varepsilon^{1/k} r_n^{-1}, \theta_n)}{V_0(r_n^{-1}, \theta_n)} & = \left(\frac{V_0(\varepsilon^{-1/k}(\varepsilon^{1/k} r_n^{-1}), \theta_n)}{V_0(\varepsilon^{1/k} r_n^{-1}, \theta_n)} \right)^{-1} \\
& \leqslant (m(\varepsilon^{-1/k})^{-1/a_1 - \delta})^{-1} \\
& = \frac{1}{m} \varepsilon^{(1/k)(-1/a_1 - \delta)}.
\end{aligned}$$

Note that n_0 does depend on ε in general.

Choose $n_1 \geqslant 1$ such that $\varepsilon^{1/k} r_n^{-1} \geqslant t_0$ for all $n \geqslant n_1$ where t_0 is as in (2.5). Then for all $n \geqslant \max(n_0, n_1)$ we get from (2.5),

$$\begin{aligned}
n r_n^{2k} & U_{2k}(\varepsilon^{1/k} r_n^{-1}, \theta_n) \\
& = \varepsilon^2 \frac{U_{2k}(\varepsilon^{1/k} r_n^{-1}, \theta_n)}{(\varepsilon^{1/k} r_n^{-1})^{2k} \, V_0(\varepsilon^{1/k} r_n^{-1}, \theta_n)} \frac{V_0(\varepsilon^{1/k} r_n^{-1}, \theta_n)}{V_0(r_n^{-1}, \theta_n)} \, n V_0(r_n^{-1}, \theta_n) \\
& \leqslant \varepsilon^2 \frac{1}{A} \frac{1}{m} \varepsilon^{-(1/k)(1/a_1 + \delta)} n(A_n \mu)\{x \colon |\langle x, b \rangle| > 1\}
\end{aligned}$$

In view of (2.1) and the portmanteau theorem (Theorem 2.1 in [1]) we get

$$\limsup_{n\to\infty} n(A_n\mu)\{|\langle x, b\rangle| > 1\} \leqslant \phi\{x: |\langle x, b\rangle| \geqslant 1\} = D < \infty. \tag{3.7}$$

Hence we have shown that for $0<\varepsilon<1$,

$$\limsup_{n\to\infty} n \int_{\|A\|\leqslant\varepsilon} \langle A, B\rangle^2 \, dL_{A_n}(T(\mu))(A) \leqslant \frac{D}{Am}\, \varepsilon^{(2-1/k)(1/a_1+\delta)}.$$

This implies (3.6) by the choice of δ for $B=T(b)$.

For the general case choose unit vectors $\{b_k: k=1, ..., m\} \subset \mathbb{R}^d$ such that $\{T(b_k): k=1, ..., m\}$ spans $\mathbb{V}$. Now for any $B\in\mathbb{V}$ we can write $B=\sum_{k=1}^m c_k T(b_k)$ and then by the Cauchy–Schwarz inequality

$$\begin{aligned}
& n\int_{\|A\|\leqslant\varepsilon} \langle A, B\rangle^2 \, dL_{A_n}(T(\mu))(A) \\
&\quad = \sum_{k,\ell=1}^{m} c_k c_\ell \cdot n \int_{\|A\|\leqslant\varepsilon} \langle A, T(b_k)\rangle \langle A, T(b_\ell)\rangle \; dL_{A_n}(T(\mu))(A) \\
&\quad \leqslant \sum_{k,\ell=1}^{m} |c_k c_\ell| \left(n \int_{\|A\|\leqslant\varepsilon} \langle A, T(b_k)\rangle^2 \, dL_{A_n}(T(\mu))(A) \right)^{1/2} \\
&\qquad \times \left(n \int_{\|A\|\leqslant\varepsilon} \langle A, T(b_\ell)\rangle^2 \, dL_{A_n}(T(\mu))(A) \right)^{1/2}.
\end{aligned}$$

Since we have already shown (3.6) for $B=T(b)$ the general case of an arbitrary $B\in\mathbb{V}$ follows from the special case considered above. This concludes the proof. ∎

In certain cases we can say more about the possible choice of the centering tensors B_n in (3.1).

Corollary 3.2. *Under the assumptions of Theorem* 3.1 *we can take*

$$B_n = \begin{cases} 0 & \text{if } \dfrac{1}{k} < a_1 \\[2ex] nL_{A_n}(E(\bigotimes^k X)) & \text{if } a_p < \dfrac{1}{k}, \end{cases}$$

where X is a $\mathbb{R}^d$-valued random vector with distribution μ.

Proof. In view of the standard convergence criteria for triangular arrays of random vectors (Theorem 2.3 in [16]) we can take

$$B_n = \int_{\|A\| \leqslant R} A \, dL_{A_n}(T(\mu))(A),$$

where $R > 0$ is arbitrary as long as $\{A \in \mathbb{V} : \|A\| = R\}$ has $T(\phi)$-measure zero, which holds for all but countable many $R > 0$.

Assume first that $1/k < a_1$. Let $b \in \mathbb{R}^d$ be a unit vector and set $B = T(b)$. Then by (3.8),

$$\begin{aligned}
|\langle B_n, T(b) \rangle| &= \left| \left\langle n \int_{\|A\| \leqslant R} A \, dL_{A_n}(T(\mu))(A), B \right\rangle \right| \\
&\leqslant n \int_{|\langle A, B \rangle| \leqslant R} |\langle A, B \rangle| \, dL_{A_n}(T(\mu))(A) \\
&= n \int_{|\langle L_{An}(T(x)), T(b) \rangle| \leqslant R} |\langle L_{A_n}(T(x)), T(b) \rangle| \, d\mu(x) \\
&= n \int_{|\langle x, A_n^* b \rangle| \leqslant R^{1/k}} |\langle x, A_n^* b \rangle|^k \, d\mu(x) \\
&= n r_n^k U_k(r_n^{-1} R^{1/k}, \theta_n),
\end{aligned}$$

where $A_n^* b = r_n \theta_n$ for some $r_n > 0$ and $\|\theta_n\| = 1$.

Choose $\delta > 0$ such that $(1 - (1/k)(1/a_1) + \delta) > 0$. Then, by (2.3) there exists a constant $m > 0$ such that for a fixed $0 < R < 1$ there exists a $n_0 \geqslant 1$ such that for all $n \geqslant n_0$,

$$\frac{V_0(R^{1/k} r_n^{-1}, \theta_n)}{V_0(r_n^{-1}, \theta_n)} \leqslant \frac{1}{m} R^{-(1/k)(1/a_1 + \delta)}.$$

Moreover, if we choose a $n_1 \geqslant 1$ such that $R^{1/k} r_n^{-1} \geqslant t_0$ for all $n \geqslant n_1$, where t_0 is as in (2.5), it follows that for $n \geqslant \max(n_0, n_1)$,

$$\begin{aligned}
& n r_n^k U_k(r_n^{-1} R^{1/k}, \theta_n) \\
&\quad = R \frac{U_k(r_n^{-1} R^{1/k}, \theta_n)}{(r_n^{-1} R^{1/k})^k \, V_0(r_n^{-1} R^{1/k}, \theta_n)} \frac{V_0(r_n^{-1} R^{1/k}, \theta_n)}{V_0(r_n^{-1}, \theta_n)} \, n V_0(r_n^{-1}, \theta_n) \\
&\quad \leqslant R \frac{1}{A} \frac{1}{m} R^{-(1/k)(1/a_1 + \delta)} n \, (A_n \mu)\{x : |\langle x, b \rangle| > 1\} \\
&\quad \leqslant C R^{(1 - 1/k)(1/a_1 + \delta)}
\end{aligned}$$

for some constant $C>0$. Hence we have shown that there exist positive constants C, δ_0 such that for $0<R<1$,

$$\limsup_{n\to\infty} |\langle B_n, T(b)\rangle| \leqslant CR^{\delta_0}. \tag{3.9}$$

Now let $B\in\mathbb{V}$ be arbitrary and write $B=\sum_{k=1}^{m} c_k T(b_k)$ for some unit vectors $b_k\in\mathbb{R}^d$. Then for $0<R<1$ we get from (3.9),

$$\limsup_{n\to\infty} |\langle B_n, B\rangle| \leqslant \sum_{k=1}^{m} |c_k| \limsup_{n\to\infty} |\langle B_n, T(b_k)\rangle| \leqslant KR^{\delta_0}$$

for some constant $K>0$ and δ_0 as above. Since $\delta_0>0$ we see that by taking R arbitrary small we can choose $B_n=0$.

In the other case we have $a_1<\cdots<a_p<1/k$. Then, for $\zeta>1/a_1$, by Lemma 2 of [10] both U_ζ and V_k are uniformly R-O varying; especially (2.3), (2.4) and (2.5) hold for some constants r_0, $t_0>0$. The existence of V_k in this case implies that $E\langle X, \theta\rangle^k$ is finite for every $\theta\in\mathbb{R}^d\backslash\{0\}$. But since $\langle E\bigotimes^k X, \bigotimes^k \theta\rangle = E\langle X, \theta\rangle^k$ it follows that $E(\bigotimes^k X)$ exists in this case.

Use (3.8) to write for $R\geqslant 1$ $B_n=nL_{A_n}(E(\bigotimes^k X))-I_n$ where

$$I_n = n\int_{\|A\|>R} A\, dL_{A_n}(T(\mu))(A).$$

Then, for a unit vector $b\in\mathbb{R}^d$ we have

$$\begin{aligned}
|\langle I_n, T(b)\rangle| &\leqslant n\int_{\|A\|>R} |\langle A, T(b)\rangle|\, dL_{A_n}(T(\mu))(A)\\
&= n\int_{|\langle A, T(b)\rangle|>R} |\langle A, T(b)\rangle|\, dL_{A_n}(T(\mu))(A)\\
&\quad + n\int_{\|A\|>R \text{ and } |\langle A, T(b)\rangle|\leqslant R} |\langle A, T(b)\rangle|\, dL_{A_n}(T(\mu))(A).
\end{aligned} \tag{3.10}$$

Now if we write $A_n^* b = r_n\theta_n$ with $r_n>0$ and $\|\theta_n\|=1$ as before, then

$$\begin{aligned}
& n\int_{|\langle A, T(b)\rangle|>R} |\langle A, T(b)\rangle|\, dL_{A_n}(T(\mu))(A)\\
&\quad = nr_n^k\, V_k(R^{1/k}r_n^{-1}, \theta_n)\\
&\quad = R^{1-\zeta/k}\, \frac{(R^{1/k}r_n^{-1})^{\zeta-k}\, V_k(R^{1/k}r_n^{-1}, \theta_n)}{U_\zeta(R^{1/k}r_n^{-1}, \theta_n)} \cdot \frac{U_\zeta(R^{1/k}r_n^{-1}, \theta_n)}{U_\zeta(r_n^{-1}, \theta_n)}\\
&\qquad \times \frac{U_\zeta(r_n^{-1}, \theta_n)}{(r_n^{-1})^\zeta\, V_0(r_n^{-1}, \theta_n)} \cdot nV_0(r_n^{-1}, \theta_n).
\end{aligned}$$

For $R \geqslant 1$ an application of (2.3)–(2.5) together with 3.7 yields

$$\limsup_{n \to \infty} n \int_{|\langle A,\, T(b)\rangle| > R} |\langle A, T(b)\rangle| \, dL_{A_n}(T(\mu))(A) \leqslant KR^{-\delta_0} \tag{3.11}$$

for some constant $K > 0$ and $\delta_0 = 1/ka_p - 1 - \delta/k > 0$ if $\delta > 0$ is chosen small enough.

To estimate the other integral on the right-hand side of 3.10 choose $\{b_l\}$ an orthonormal basis of $\mathbb{R}^d$ and note that if $\|y\|^k > R$ then $|\langle y, b_l\rangle|^k > R/d^{k/2}$ for some $l = 1, ..., d$. Hence

$$\begin{aligned}
n \int_{\|A\| > R \text{ and } |\langle A,\, T(b)\rangle| \leqslant R} & |\langle A, T(b)\rangle| \, dL_{A_n}(T(\mu))(A) \\
&\leqslant Rn \int_{\|A\| > R} dL_{A_n}(T(\mu))(A) \\
&= Rn \int_{\|A_n x\|^k > R} d\mu(x) \\
&\leqslant R \sum_{l=1}^{d} n \int_{|\langle A_n x,\, b_l\rangle|^k > R/d^{k/2}} d\mu(x) \\
&= \sum_{l=1}^{d} RnV_0\left((R^{1/k}/\sqrt{d})\, r_{nl}^{-1}, \theta_{nl}\right),
\end{aligned}$$

where $A_n^* b_l = r_{nl}\theta_{nl}$ with $r_{nl} > 0$ and $\|\theta_{nl}\| = 1$.

Apply (2.3)–(2.5) and (3.7) to see that each individual summand in the sum above can be bounded for all large n by

$$\begin{aligned}
&RnV_0((R^{1/k}/\sqrt{d})\, r_{nl}^{-1}, \theta_{nl}) \\
&\quad = d^{\zeta/2} R^{1-\zeta/k} \frac{((R^{1/k}/\sqrt{d})\, r_{nl}^{-1})^{\zeta}\, V_0((R^{1/k}/\sqrt{d})\, r_{nl}^{-1}, \theta_{nl})}{U_\zeta((R^{1/k}/\sqrt{d})\, r_{nl}^{-1}, \theta_{nl})} \\
&\quad\quad \times \frac{U_\zeta((R^{1/k}/\sqrt{d})\, r_{nl}^{-1}, \theta_{nl})}{U_\zeta(r_{nl}^{-1}, \theta_{nl})} \cdot \frac{U_\zeta(r_{nl}^{-1}, \theta_{nl})}{(r_{nl}^{-1})^{\zeta}\, V_0(r_{nl}^{-1}, \theta_{nl})} \cdot nV_0(r_{nl}^{-1}, \theta_{nl}) \\
&\quad \leqslant KR^{1 - 1/ka_p - \delta/k} = KR^{-\delta_0}
\end{aligned}$$

for some positive constant K, where $\delta_0 = 1/ka_p - 1 - \delta/k > 0$ as above. Hence, in view of (3.10) and (3.11) we have shown that for some positive real constant C_1 and all $R \geqslant 1$,

$$\limsup_{n \to \infty} |\langle I_n, T(b)\rangle| \leqslant C_1 R^{-\delta_0}$$

for every unit vector $b \in \mathbb{R}^d$, where $\delta_0 > 0$.

This implies as before that

$$\limsup_{n\to\infty} |\langle I_n, B\rangle| \leqslant C_2 R^{-\delta_0}$$

for every unit vector $B \in \mathbb{V}$ and some constant $C_2 > 0$. Taking $R > 1$ arbitrary large, we see that the convergence in (3.1) still holds when $B_n = nL_{A_n}(E(\bigotimes^k X))$. This concludes the proof. ∎

After calculating the asymptotic distribution of the normalized sum of i.i.d. symmetric k-tensors $\bigotimes^k X_i$ in Theorem 3.1 above we now investigate the structure of the limit W in (3.1).

PROPOSITION 3.3. *Under the assumptions of Theorem* 3.1 *the limit* W *with Lévy representation* $[C, 0, T(\phi)]$ *is a full operator stable law on a certain subspace of* $\mathbb{U}$ *of* $\mathbb{V}$.

Proof. The Lévy measure $T(\phi)$ of W satisfies by (2.2) the equation

$$L_{t^E}(T(\phi)) = T(t^E\phi) = t \cdot T(\phi) \qquad \text{for all} \quad t > 0.$$

Since W has no normal component Proposition 4.3.2 of [5] implies that W is full $(L_{t^E})_{t>0}$-operator stable on the subspace $\mathbb{U} = \mathrm{span}(\mathrm{supp}(\Psi))$ where Ψ denotes the distribution of W. Here $\mathrm{supp}\,\rho$ denotes the support of a measure ρ and $\mathrm{span}(S)$ is the linear subspace generated by the vectors in S. Note that $(L_{t^E})_{t>0}$ is a one-parameter subgroup of $\mathrm{GL}(\mathbb{V})$. ∎

THEOREM 3.4. *Let* W *be as in Theorem* 3.1. *Then for all* $\theta \in \mathbb{R}^d \backslash \{0\}$ *the random variable* $\langle W, \bigotimes^k \theta\rangle$ *has a density with respect to Lebesgue measure on* $\mathbb{R}$.

Proof. Fix any $\theta \in \mathbb{R}^d \backslash \{0\}$ and let ρ denote the distribution of $\langle W, \bigotimes^k \theta\rangle$. Using the Lévy representation of W we get for the Fourier transform $\hat{\rho}$ of ρ;

$$\begin{aligned}\hat{\rho}(s) &= E\left[e^{is\langle W, \otimes^k \theta\rangle}\right] = E\left[e^{i\langle W, s\otimes^k \theta\rangle}\right] \\ &= \exp\left\{ i\langle C, s \bigotimes^k \theta\rangle + \int_{\mathbb{V}\backslash\{0\}} \left(e^{i\langle M, s\,\otimes^k \theta\rangle} - 1 - \frac{i\langle M, s \bigotimes^k \theta\rangle}{1 + \|M\|^2} \right) \right. \\ &\qquad \left. \times\, dT(\phi)(M) \right\}\end{aligned}$$

$$= \exp\left\{ is\langle C, \otimes^k \theta\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{is\langle \otimes^k x,\, \otimes^k \theta\rangle} - 1 \right.\right.$$

$$\left.\left. - \frac{is\langle \otimes^k x, \otimes^k \theta\rangle}{1+\|\otimes^k x\|^2}\right) d\phi(x)\right\}$$

$$= \exp\left\{ is\langle C, \otimes^k \theta\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{is\langle x, \theta\rangle^k} - 1 - \frac{is\langle x, \theta\rangle^k}{1+\|x\|^{2k}}\right) d\phi(x)\right\}$$

and hence

$$\begin{aligned} |\hat{\rho}(s)| &= (\hat{\rho}(s)\,\overline{\hat{\rho}(s)})^{1/2} \\ &= \exp\left\{ \int_{\mathbb{R}^d\setminus\{0\}} [\cos(s\langle x, \theta\rangle^k) - 1]\, d\phi(x)\right\} \\ &= \exp\{-u(s)\}, \end{aligned}$$

where

$$u(s) = \int_{\mathbb{R}^d\setminus\{0\}} [1 - \cos(s\langle x, \theta\rangle^k)]\, d\phi(x). \tag{3.12}$$

Note that $u(s) = u(-s)$ so it is enough to consider $s > 0$. Now write $s^{1/k}\theta = t^{E^*}\bar{\theta}$ for some $\|\bar{\theta}\| = 1$ and some $t = t(s) > 0$. Since $0 < a_1 < \cdots < a_p$ denote the real parts of the eigenvalues of E we have $t(s) \to \infty$ as $s \to \infty$. Hence there exists a $s_0 = s_0(\theta)$ such that $t(s) \geqslant 1$ for all $s \geqslant s_0$. Furthermore, the results in [4, chapter 6] imply that for $\varepsilon > 0$ there exists a constant $B > 0$ with $\|t^{E^*}\| \leqslant Bt^{a_p+\varepsilon}$ for all $t \geqslant 1$. Then we get for $s \geqslant s_0$,

$$s^{1/k}\,\|\theta\| = \|t^{E^*}\bar{\theta}\| \leqslant \|t^{E^*}\| \leqslant Bt^{a_p+\varepsilon},$$

which implies $s^{1/k} \leqslant B(\theta)\, t^{a_p+\varepsilon}$ for some constant $B(\theta) > 0$. Hence for some $\delta = \delta(\varepsilon) > 0$

$$t = t(s) \geqslant \bar{B}(\theta)\, s^{1/ka_p - \delta} \tag{3.13}$$

for some constant $\bar{B}(\theta) > 0$ and all $s \geqslant s_0$. Note that $\delta(\varepsilon) > 0$ can be made arbitrary small if $\varepsilon > 0$ is chosen small enough.

Using (3.12), (3.13), and 2.2 we then get for all $s \geqslant s_0$,

$$\begin{aligned} u(s) &= \int_{\mathbb{R}^d\setminus\{0\}} [1 - \cos(\langle x, s^{1/k}\theta\rangle^k)]\, d\phi(x) \\ &= \int_{\mathbb{R}^d\setminus\{0\}} [1 - \cos(\langle x, t^{E^*}\bar{\theta}\rangle^k)]\, d\phi(x) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d\setminus\{0\}} [1-\cos(\langle t^E x, \bar{\theta}\rangle^k)]\, d\phi(x) \\
&= \int_{\mathbb{R}^d\setminus\{0\}} [1-\cos(\langle x, \bar{\theta}\rangle^k)]\, d(t^E\phi)(x) \\
&= t\cdot \int_{\mathbb{R}^d\setminus\{0\}} [1-\cos(\langle x, \bar{\theta}\rangle^k)]\, d\phi(x) \\
&\geqslant \bar{B}(\theta)\, s^{1/ka_p-\delta} K(\bar{\theta}),
\end{aligned}
$$

where for any $\|\theta\|=1$,

$$
K(\theta) = \int_{\mathbb{R}^d\setminus\{0\}} [1-\cos(\langle x, \theta\rangle^k)]\, d\phi(x).
$$

Then K is continuous and $K(\theta)\geqslant 0$ for all $\|\theta\|=1$.

In fact, if θ_n is any sequence of unit vectors and $\theta_n\to\theta$ as $n\to\infty$, then the integrand $f_n(x)=1-\cos(\langle x, \theta_n\rangle^k)\to f(x)=1-\cos(\langle x, \theta\rangle^k)$ as $n\to\infty$ for all $x\in\mathbb{R}^d$. Furthermore, for some constant $C>0$

$$
\begin{aligned}
|f_n(x)| &= |f_n(x)|\, I(\|x\|\leqslant 1) + |f_n(x)|\, I(\|x\|>1) \\
&\leqslant C\|\mathrm{x}\|^{2k}\, I(\|x\|\leqslant 1) + 2I(\|x\|>1)
\end{aligned}
$$

and the right-hand side is by (3.3) integrable with respect to ϕ on $\mathbb{R}^d\setminus\{0\}$. Then Lebesgue's dominated convergence theorem implies $K(\theta_n)\to K(\theta)$ as $n\to\infty$ proving the continuity of K.

We will show below that $K(\theta)\neq 0$ for all $\|\theta\|=1$ and hence

$$
K_0 = \inf_{\|\theta\|=1} K(\theta) > 0,
$$

which implies that $u(s)\geqslant As^{1/(ka_p)-\delta}$ for all $s\geqslant s_0$ and some constant $A>0$.

Hence

$$
|\hat{\rho}(s)| \leqslant \begin{cases} 1 & \text{for} \quad |s|<s_0 \\ \exp(-A\,|s|^{1/ka_p-\delta} & \text{for} \quad |s|\geqslant s_0 \end{cases}
$$

so $\int_{\mathbb{R}} |\hat{\rho}(s)|\, ds<\infty$ and therefore by the Fourier inversion formula ρ has a Lebesgue density.

It remains to show that $K(\theta)\neq 0$ for all $\|\theta\|=1$. Let us assume that there exists a $\|\theta_0\|=1$ with $K(\theta_0)=0$. Then

$$
K(\theta_0) = \int_{\mathbb{R}^d\setminus\{0\}} [1-\cos(\langle x, \theta_0\rangle^k)]\, d\phi(x) = 0
$$

implies $\cos(\langle x, \theta_0\rangle^k)=1$ ϕ-almost everywhere. Now let $L_j=\{x \in \mathbb{R}^d: |\langle x, \theta_0\rangle|=(2\pi j)^{1/k}\}$ for $j \geqslant 0$. Then we get

$$\operatorname{supp}(\phi) \subset \bigcup_{j=0}^{\infty} L_j.$$

But since by (2.2) $t^E\phi = t \cdot \phi$ for all $t>0$ we have $\operatorname{supp}(t^E\phi)=\operatorname{supp}(\phi)$ for all $t>0$. So if $x \in \operatorname{supp}(\phi)$ then $t^E x \in \operatorname{supp}(\phi)$ and hence $t^E x \in \bigcup_{j=0}^{\infty} L_j$ for all $t>0$. Furthermore, since $t^E x \to 0$ as $t \to 0$ we get $|\langle t^E x, \theta_0\rangle| < \varepsilon$ for all $t \leqslant t_0 = t_0(x, \varepsilon)$. Hence if $0<\varepsilon<(2\pi)^{1/k}$ we have $t^E x \in L_0$ for all $t \leqslant t_0$.

Since ϕ is not supported on any proper subspace of $\mathbb{R}^d$ there exist $x_1, ..., x_d \in \operatorname{supp}(\phi)$ linearly independent and a $t_1>0$ such that $t_1^E x_i \in L_0$ for $i=1, ..., d$. Note that since t_1^E is invertible the vectors $t_1^E x_1, ..., t_1^E x_d$ are linearly independent. Hence $\dim L_0 = d$ and then $L_0=\mathbb{R}^d$. On the other hand, $\theta_0 \notin L_0$ since $|\langle\theta_0, \theta_0\rangle| = \|\theta_0\| \neq 0$ which is a contradiction. This concludes the proof. ∎

4. APPLICATIONS

In this section we apply the results of Section 3 to construct an estimate of the value a_p for a regularly varying measure $\mu \in \mathrm{RV}(E)$. Since from Lemma 2 of [10] it follows that V_ρ exists for all $\rho < 1/a_p$, we have $E\|X\|^\rho < \infty$ for all $\rho < 1/a_p$. Hence the value of a_p gives direct information about the moments of a random vector with distribution μ. Furthermore we show that certain polynomials in the components of $X_1, ..., X_n$ are, after a suitable normalization, stochastically compact.

Let $\mu \in \mathrm{RV}(E)$ and let $0<a_1<\cdots<a_p$ be the real parts of the eigenvalues of E. Assume that $X_1, X_2, ...$ are i.i.d. according to μ and choose an *even* natural number $k \geqslant 2$ with $1/a_1 < k$. Consider the symmetric k-tensor

$$M_n = \sum_{i=1}^{n} \otimes^k X_i.$$

Then we know from Corollary 3.2 that

$$L_{A_n}(M_n) \Rightarrow W,$$

where W is infinitely divisible with Lévy representation $[C, 0, T(\phi)]$ and A_n, ϕ are as in (2.1). Note that for $\|\theta\|=1$ $\langle M_n, \otimes^k \theta\rangle = \sum_{i=1}^n \langle X_i, \theta\rangle^k \geqslant 0$ since k is even. Consider the random variable

$$\lambda_n = \max_{\|\theta\|=1} \langle M_n, \otimes^k \theta\rangle$$

and let

$$\hat{a}_n = \frac{\log^+ \lambda_n}{k \log n}$$

where $\log^+ x = \max\{0, \log x\}$. This estimator is a partial generalization of the moment estimator for generalized domains of attraction considered in [12], where we had $k=2$ and $X_1 \in \mathrm{GDOA}(Y)$. In the more general case we get:

THEOREM 4.1. *Under the assumptions above we have*

$$\hat{a}_n \to a_p \qquad \textit{in probability.}$$

Proof. For $\delta > 0$ arbitrary we have

$$\begin{aligned}
&P\{|\hat{a}_n - a_p| > \delta\} \\
&\quad \leqslant P\{\log^+ \lambda_n > k(a_p+\delta) \log n\} + P\{\log^+ \lambda_n < k(a_p - \delta) \log n\} \\
&\quad = P\left\{ \max_{\|\theta\|=1} \langle M_n, \bigotimes^k \theta \rangle > n^{k(a_p+\delta)} \right\} \\
&\qquad + P\left\{ \max_{\|\theta\|=1} \langle M_n, \bigotimes^k \theta \rangle < n^{k(a_p-\delta)} \right\} \\
&\quad = I_1^{(n)} + I_2^{(n)}.
\end{aligned}$$

We now consider $I_1^{(n)}$ and $I_2^{(n)}$ separately. Choose $\rho > 0$ such that $ka_p < 1/\rho < k(a_p+\delta)$ and hence $k\rho < 1/a_p$. Note that from the existence of $V_{k\rho}(t, \theta)$ for all $\|\theta\| = 1$ by Lemma 2 of [10] it follows that $E\,\|X_1\|^{k\rho} < \infty$. Since $1/k < a_1 < a_p$ we have $0 < \rho < 1$ and hence $|x+y|^\rho \leqslant |x|^\rho + |y|^\rho$ for all $x, y \in \mathbb{R}$. Then, by Markov's inequality

$$\begin{aligned}
I_1^{(n)} &= P\left\{ \max_{\|\theta\|=1} \sum_{i=1}^n \langle \bigotimes^k X_i, \bigotimes^k \theta \rangle > n^{k(a_p+\delta)} \right\} \\
&= P\left\{ \max_{\|\theta\|=1} \sum_{i=1}^n \langle X_i, \theta \rangle^k > n^{k(a_p+\delta)} \right\} \\
&\leqslant P\left\{ \sum_{i=1}^n \|X_i\|^k > n^{k(a_p+\delta)} \right\} \\
&\leqslant \left(\frac{1}{n^{k(a_p+\delta)}} \right)^\rho E\left| \sum_{i=1}^n \|X_i\|^k \right|^\rho \\
&\leqslant n^{1-k\rho(a_p+\delta)}\, E\,\|X_1\|^{k\rho},
\end{aligned}$$

which tends to zero as $n \to \infty$ by the choice of ρ.

For $I_2^{(n)}$ we need the spectral decomposition of multivariable regular variation as in [13, Section 2]. Since $A_{[\lambda n]} A_n^{-1} \to \lambda^{-E}$ as $n \to \infty$ we get $(A^*_{[\lambda n]})^{-1}((A^*_n)^{-1})^{-1} \to \lambda^{E^*}$ so $((A^*_n)^{-1})_n$ is a regularly varying sequence with index E^* in the sense of [13]. Now let $L_1 \subset L_2 \cdots \subset L_p = \mathbb{R}^d$ be the nested sequence of subspaces constructed in Lemma 2.3 of [13] for the function $f(t) = (A^*_{[t]})^{-1}$. For a unit vector $\theta_0 \in L_p \backslash L_{p-1}$ we write $(A^*_n)^{-1} \theta_0 = r_n \theta_n$ for some $r_n > 0$ and $\|\theta_n\| = 1$. Then it follows from Lemma 2.3 of [13] that for any $\varepsilon > 0$ there exists a $n_0 \geqslant 1$ such that $n^{a_p - \varepsilon} \leqslant r_n \leqslant n^{a_p + \varepsilon}$ for all $n \geqslant n_0$. We choose $\varepsilon < \delta$. Then we get for $n \geqslant n_0$,

$$
\begin{aligned}
I_2^{(n)} &= P\left\{\max_{\|\theta\|=1} \langle M_n, \otimes^k \theta\rangle < n^{k(a_p - \delta)}\right\} \\
&\leqslant P\left\{\langle M_n, \otimes^k \theta_0\rangle < n^{k(a_p-\delta)}\right\} \\
&= P\left\{\langle L_{A_n}(M_n), L_{(A_n^*)^{-1}}(\otimes^k \theta_0\rangle < n^{k(a_p-\delta)}\right\} \\
&= P\left\{\langle L_{A_n}(M_n), \otimes^k ((A_n^*)^{-1}\theta_0)\rangle < n^{k(a_p-\delta)}\right\} \\
&= P\left\{\langle L_{A_n}(M_n), \otimes^k \theta_n\rangle < r_n^{-k} n^{k(a_p-\delta)}\right\} \\
&\leqslant P\left\{\langle L_{A_n}(M_n), \otimes^k \theta_n\rangle < n^{k(\varepsilon-\delta)}\right\}.
\end{aligned}
$$

Since $\|\theta_n\| = 1$, given any sequence (n_l) of natural numbers there exists a further sequence $(n') \subset (n_l)$ along which $\theta_{n'} \to \theta$ where $\|\theta\| = 1$. Then by Theorem 5.5 of [1] and Corollary 3.2, $\langle L_{A_n}(M_n), \otimes^k \theta_n\rangle \Rightarrow \langle W, \otimes^k \theta\rangle$ along (n'). Note that since k is even $\langle L_{A_n}(M_n), \otimes^k \theta_n\rangle = \sum_{i=1}^n \langle A_n X_i, \theta_n\rangle^k \geqslant 0$ almost surely and hence $\langle W, \otimes^k \theta\rangle \geqslant 0$ almost surely. Hence the density of $\langle W, \otimes^k \theta\rangle$ by Theorem 3.4 is supported in $[0, \infty)$. Then for any $\varepsilon_1 > 0$ there exists a $R > 0$ such that $P\{\langle W, \otimes^k \theta\rangle < R\} < \varepsilon_1/2$. Now choose a $n_1 \geqslant n_0$ such that $(n')^{k(\varepsilon-\delta)} < R$ and

$$
|P\{\langle L_{A_{n'}}(M_{n'}), \otimes^k \theta_{n'}\rangle < R\} - P\{\langle W, \otimes^k \theta\rangle < R\}| < \varepsilon_1/2
$$

for all $n' \geqslant n_1$. Then for all $n' \geqslant n_1$,

$$
\begin{aligned}
P\left\{\langle L_{A_{n'}}(M_{n'}), \otimes^k \theta_{n'}\rangle < (n')^{k(\varepsilon-\delta)}\right\} &\leqslant P\left\{\langle L_{A_{n'}}(M_{n'}), \otimes^k \theta_{n'}\rangle < R\right\} \\
&\leqslant P\{\langle W, \otimes^k \theta\rangle < R\} + \varepsilon_1/2 < \varepsilon_1,
\end{aligned}
$$

showing that $I_2^{(n)} \to 0$ along (n'). Since any sequence has a further subsequence with the property $I_2^{(n)} \to 0$ as $n \to \infty$ which concludes the proof. ∎

As another application of the limit theorems in Section 3 we now consider the asymptotic behavior of sums of certain homogeneous polynomials in the components of the X_i. Our results are formulated in terms of stochastic compactness.

Recall that a sequence $(Z_n)_n$ of random variables on the real line is called *stochastically compact*, if the corresponding sequence of distributions is weakly relatively compact and all limit points are nondegenerate.

THEOREM 4.2. *Let $\mu \in \mathrm{RV}(E)$ where $0 < a_1 < \cdots < a_p$ are the real parts of the eigenvalues of E. Choose any natural $k \geqslant 1$ such that $1/(2k) < a_1$. Then for all nonzero $\theta \in \mathbb{R}^d$ there exists a sequence $(r_n)_n$ of positive real numbers tending to zero and a sequence of shifts $(s_n)_n \subset \mathbb{R}$ such that*

$$\left(r_n^k \sum_{i=1}^{n} \langle \otimes^k X_i, \otimes^k \theta \rangle - s_n \right)_n \tag{4.1}$$

is stochastically compact with limit set contained in the set

$$\{ \langle W, \otimes^k \theta_0 \rangle : \|\theta_0\| = 1 \},$$

where W is the limit in (3.1).

Proof. For $\theta \in \mathbb{R}^d \backslash \{0\}$ let

$$r_n \, \|(A_n^*)^{-1} \theta\|^{-1},$$

where A_n is as in (2.1) Note that $((A_n^*)^{-1})_n$ is a regularly varying sequence with index E^* and hence by Lemma 2.3 of [13], $\|(A_n^*)^{-1} \theta\| \to \infty$ as $n \to \infty$. Write $(A_n^*)^{-1} \theta = r_n^{-1} \theta_n$ for some $\|\theta_n\| = 1$. But

$$\begin{aligned} \langle \otimes^k X_i, \otimes^k \theta \rangle &= \langle L_{A_n}(\otimes^k X_i), L_{(A_n^*)^{-1}}(\otimes^k \theta) \rangle \\ &= \langle L_{A_n}(\otimes^k X_i), \otimes^k ((A_n^*)^{-1} \theta) \rangle \\ &= r_n^{-k} \langle L_{A_n}(\otimes^k \theta), \otimes^k \theta_n \rangle. \end{aligned}$$

Hence, if we set $s_n = \langle B_n, \otimes^k \theta_n \rangle$ where B_n is as in (3.1),

$$\begin{aligned} & r_n^k \sum_{i=1}^{n} \langle \otimes^k X_i, \otimes^k \theta \rangle - s_n \\ &\qquad = \sum_{i=1}^{n} \langle L_{A_n}(\otimes^k X_i), \otimes^k \theta_n \rangle - \langle B_n, \otimes^k \theta_n \rangle \\ &\qquad = \left\langle L_{A_n}\left(\sum_{i=1}^{n} \otimes^k X_i \right) - B_n, \otimes^k \theta_n \right\rangle. \end{aligned}$$

Now any subsequence contains a further subsequence (n') such that $\theta_n \to \theta_0$ along (n') for some $\|\theta_0\| = 1$. Now Theorem 5.5 of [1] together with (3.1) implies

$$r_n^k \sum_{i=1}^{n} \langle \otimes^k X_i, \otimes^k \theta \rangle - s_n \Rightarrow \langle W, \otimes^k \theta_0 \rangle$$

along (n'). Note that by Theorem 3.4, $\langle W, \otimes^k \theta_0 \rangle$ has a density on $\mathbb{R}$ and hence is nondegenerate. ∎

Remark 4.3. The mapping $\mathbb{R}^d \ni x \mapsto \langle \otimes^k x, \otimes^k \theta \rangle = \langle x, \theta \rangle^k$ is a homogeneous polynomial of degree k in the components of x. Hence (4.1) is an i.i.d. sum of certain homogeneous polynomials in the components of the X_i.

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